1 Introduction

Pareto Dominates Feasible alternative A *Pareto-dominates* alternative B if nobody prefers B to A and at least on person prefers A to B.

Pareto Efficient Feasible alternative A is *Pareto efficient* if no feasible alternative Pareto-dominates A. We cannot make anyone better off without harming at least one person.

x = a good or service
m = numeraire (a proxy for all goods other than good X). Can be used for consumption or as an input for good X
ω = initial endowment of numeraire

$C(x_B, x_B)$ = physical cost function. The amount of numeraire needed to produce a given amount of good X

Feasible Allocation A list of four numbers x_B, m_B, x_R, m_R satisfying $m_B + m_R + C(x_B + x_R) \leq \omega$.

Production Efficiency $m_B + m_R + C(x_B + x_R) = \omega$.

Utility Functions

Quasilinear preferences:
$U_i(x_i, m_i) = \sqrt{v_i(x_i) + m_i} \quad i = B, R$

$v_B(x_B)$ = The (total) *valuation* by Blue of x_B units of good X. How much good x_B is worth to Blue, in units of numeraire
$v_B(0) = 0$
$v'_B(x_B)$ = The *marginal valuation* of x_i in units of numeraire, or the *inverse demand function*

Social Surplus The sum of valuations − cost.
$$S(x_B, x_R) = v_B(x_B) + v_R(x_R) - C(x_B + x_R)$$

Fact: A feasible allocation is *efficient* if and only if:
(1) $m_B + m_R + C(x_B + x_R) = \omega$
(2) (x_B, x_R) maximizes social surplus

Deadweight Loss The difference between the actual social surplus and the maximum social surplus.

2 Private goods without externalities

2.1 Maximization of Social Surplus

$$\max_{x_B, x_R} S(x_B, x_R) = v_B(x_B) + v_R(x_R) - C(x_B + x_R)$$

$$(1)\ \frac{\partial S}{\partial x_B} = 0 \Rightarrow v_B'(x_B^*) = C'(x_B^* + x_R^*)$$

$$(2)\ \frac{\partial S}{\partial x_R} = 0 \Rightarrow v_R'(x_R^*) = C'(x_B^* + x_R^*)$$

$$\Rightarrow \underbrace{v_B'(x_B^*) = v_R'(x_R^*)}_{\text{marginal valuation}} = \underbrace{C'(x_B^* + x_R^*)}_{\text{marginal social cost}}$$

The marginal valuation of X for each agent (in units of numeraire) must be equal to the marginal cost (in units of numeraire) of producing an additional unit of X.

Utility Minifrontier $[U_B]^2 + [U_R]^2 = S(x_B, x_R) + \omega$. A circle with radius $\sqrt{S(x_B, x_R) + \omega}$.

First Fundamental Theorem of Welfare Economics

Under the following conditions:
(1) Absence of externalities
(2) Absence of public goods
(3) Symmetrical (not necessarily perfect) information,
the allocation at a general competitive equilibrium is *efficient*.

Efficient $\neq$ equitable

2.2 Free Market Equilibrium

p = price of x_B
1 = price of m_B
ω_B = initial wealth

2.2.1 UMAX

Maximize $U_B(x_B, m_B) = \sqrt{v_B(x_B) + m_B}$ subject to $px_B + 1 \cdot m_B = \omega_B$

$$\max_{x_B} U_B(x_B) = v_B(x_B) + \omega_B - px_B$$

$$(1)\ \frac{\partial U_B}{\partial x_B} = 0 \Rightarrow \underbrace{v_B'(\tilde{x}_B)}_{\text{marginal valuation}} = \underbrace{p}_{\text{marginal private cost}}$$ **Inverse Demand Function**

The marginal valuation of X (in units of numeraire) must be equal to the units of numeraire given up in order to buy an additional unit of X.

2.2.2 π MAX

$$\max_y \pi(y) = py - C(y)$$

$$(1)\ \frac{\partial \pi}{\partial y} = 0 \Rightarrow \underbrace{C'(y)}_{\text{marginal cost}} = \underbrace{p}_{\text{marginal revenue}}$$

2.2.3 Competitive Equilibrium

At a competitive equilibrium, the aggregate output of firms equals the aggregate demand by consumers: $\tilde{x}_B + \tilde{x}_R = y$.
$\Rightarrow v'_B(\tilde{x}_B) = v'_R(\tilde{x}_R) = C'(\tilde{x}_B + \tilde{x}_R) = p$

Total Valuation of $\bar{x}$ The area under the demand curve up to $\bar{x}$.
$\int_0^{\bar{x}} v'(x)\,\mathrm{d}x = v(\bar{x})$.

Total Cost of $\bar{x}$ The area under the cost curve up to $\bar{x}$ (in the absence of fixed costs). $\int_0^{\bar{x}} C'(x)\,\mathrm{d}x + F = C_v(\bar{x}) + F$.

Social Surplus The area below the aggregate demand curve but above the aggregate supply curve (in the absence of fixed costs).
Consumer surplus + profits.

3 Externalities

Externality When the utility function of a consumer (or the production function of a firm) includes consumption (or production) variables whose values are chosen by other consumers (or firms) without particular attention to the recipient of the externality.

Unidirectional Externalities

$U_B(x_B, m_B) = \sqrt{v_B(x_B) + m_B}$
$U_R(x_R, x_B, m_R) = \sqrt{v_R(x_R) - \gamma(Z) + m_R}$
Z = amount of externality $\Rightarrow Z = x_B$
$\gamma(Z)$ = external damage (or cost) in units of numeraire

$S(x_B, x_R) = v_B(x_B) + v_R(x_R) - \gamma(x_B) - C(x_B + x_R)$

3.1 Maximization of Social Surplus

$$\max_{x_B, x_R} S(x_B, x_R) = v_B(x_B) + v_R(x_R) - \gamma(x_B) - C(x_B + x_R)$$

$$(1)\ \frac{\partial S}{\partial x_B} = 0 \Rightarrow \underbrace{v_B'(x_B^*)}_{\text{marginal social benefit}} = \underbrace{\underbrace{C'(x_B^* + x_R^*)}_{\text{marginal production cost}} + \underbrace{\gamma'(x_B^*)}_{\text{marginal external cost}}}_{\text{marginal social cost}}$$

$$(2)\ \frac{\partial S}{\partial x_R} = 0 \Rightarrow \underbrace{v_R'(x_R^*)}_{\text{marginal social benefit}} = \underbrace{C'(x_B^* + x_R^*)}_{\text{marginal social cost}}$$

3.2 Free Market Equilibrium

3.2.1 UMAX Blue

Maximize $U_B(x_B, m_B) = \sqrt{v_B(x_B) + m_B}$ subject to $px_B + m_B = \omega_B$

$$\max_{x_B} v_B(x_B) + \omega_B - px_B$$

$$(1)\ \frac{\partial U_B}{\partial x_B} = 0 \Rightarrow \underbrace{v_B'(\tilde{x}_B)}_{\text{marginal internal benefit}} = \underbrace{p}_{\text{marginal internal cost}}$$

$\tilde{m}_B = \omega_B - p\tilde{x}_B$

3.2.2 UMAX Red

Maximize $U_R(x_R, m_R) = \sqrt{v_R(x_R) - \gamma(Z) + m_R}$ subject to $px_R + m_R = \omega_R$

$$\max_{x_R} v_R(x_R) - \gamma(Z) + \omega_R - px_R$$

$$(1)\ \frac{\partial U_R}{\partial x_R} = 0 \Rightarrow \underbrace{v_R'(\tilde{x}_R)}_{\text{marginal internal benefit}} = \underbrace{p}_{\text{marginal internal cost}}$$

$\tilde{m}_R = \omega_R - p\tilde{x}_R$

3.2.3 π MAX

$$\max_{y} \pi(y) = py - C(y)$$

$$(1)\frac{\partial \pi}{\partial y} = 0 \Rightarrow \underbrace{p}_{\text{marginal revenue}} = \underbrace{C'(y)}_{\text{marginal cost}}$$

3.2.4 Market Clearing

$\underbrace{y}_{\text{supply}} = \underbrace{\tilde{x}_B + \tilde{x}_R}_{\text{demand}}$

3.2.5 Production Efficiency

$\tilde{m}_B + \tilde{m}_R + C(y) = \omega$

If Blue imposes a *negative* externality, then the amount of private good demanded by Blue is greater than the socially optimal quantity.
$\Rightarrow \tilde{x}_B > x_B^*$

If Blue imposes a *positive* externality, then the amount of private good demanded by Blue is less than the socially optimal quantity.
$\Rightarrow \tilde{x}_B < x_B^*$

3.3 Quota

$\bar{Z}$ = max amount of x_B Blue can consume
Set $\bar{Z} = x_B^*$ to achieve efficiency.

3.4 Pigouvian Tax

Charge t for each unit of x_B consumed.

3.4.1 UMAX Blue

Maximize $U_B(x_B, m_B) = \sqrt{v_B(x_B) + m_B}$ subject to $px_B + tx_B + m_B = \omega_B$

$$\max_{x_B} v_B(x_B) + \omega_B - px_B - tx_B$$

$$(1)\ \frac{\partial U_B}{\partial x_B} = 0 \Rightarrow \underbrace{v_B'(\tilde{x}_B)}_{\text{marginal internal benefit}} = \underbrace{p + t}_{\text{marginal internal cost}}$$

Efficient if $t^* = \gamma'(x_B^*)$

3.5 Cap and Trade

$\bar{z}$ = number of permits issued
$\bar{z}_B$ = initial permits issued to B
$\bar{z} = \bar{z}_B + \bar{z}_R$
r = secondary price of permit

3.5.1 UMAX Blue

Maximize $U_B(x_B, m_B) = \sqrt{v_B(x_B) + m_B}$ subject to
$px_B + r[x_B - \bar{z}_B] + m_B = \omega_B$

$$\max_{x_B} v_B(x_B) + \omega_B - px_B - r[x_B - \bar{z}_B]$$

$$(1)\ \frac{\partial U_B}{\partial x_B} = 0 \Rightarrow \underbrace{v_B'(\tilde{x}_B)}_{\text{marginal internal benefit}} = \underbrace{p + r}_{\text{marginal internal cost}}$$

Efficient if $r^* = \gamma'(x_B^*)$

Omnidirectional Externalities

I_N = number of residents in the North $\quad i = 1, \ldots, I_N$
I_S = number of residents in the South $\quad j = I_N + 1, \ldots, I_N + I_S$
$Z = \sum_{i=1}^{I_N} x_i + \sum_{j=I_N+1}^{I_N+I_S} x_j$
$U_N(x_h, m_h, Z) = \sqrt{v_N(x_h) - \gamma_N(Z) + m_h}$
$U_S(x_k, m_k, Z) = \sqrt{v_S(x_k) - \gamma_S(Z) + m_k}$

3.6 Free Market Equilibrium

3.6.1 UMAX North

Maximize $U_N(x_h, m_h) = \sqrt{v_N(x_h) - \gamma_N(x_h + Z_{-h}) + m_h}$ subject to
$px_h + m_h = \omega_N$

$$\max_{x_h} v_N(x_h) - \gamma_N(x_h + Z_{-h}) + \omega_N - px_h$$

$$(1)\ \frac{\partial U_N}{\partial x_h} = 0 \Rightarrow \underbrace{v_N'(\tilde{x}_h)}_{\text{marginal internal benefit}} = \underbrace{p + \gamma_N'(\tilde{x}_h + Z_{-h})}_{\text{marginal internal cost}}$$

$\tilde{m}_h = \omega_N - p\tilde{x}_h$

3.6.2 UMAX South

Maximize $U_S(x_k, m_k) = \sqrt{v_S(x_k) - \gamma_S(x_k + Z_{-k}) + m_k}$ subject to
$px_k + m_k = \omega_S$

$$\max_{x_k} v_S(x_k) - \gamma_S(x_k + Z_{-k}) + \omega_S - px_k$$

$$(1)\ \frac{\partial U_S}{\partial x_k} = 0 \Rightarrow \underbrace{v_S'(\tilde{x}_k)}_{\text{marginal internal benefit}} = \underbrace{p + \gamma_S'(\tilde{x}_k + Z_{-k})}_{\text{marginal internal cost}}$$

$\tilde{m}_k = \omega_S - p\tilde{x}_k$

3.7 Nationalistic Solution

$$\max_{x_N} S_N(x_N) = I_N[v_N(x_N) - C(x_N) - \gamma_N(I_N x_N + Z_S)]$$

$$(1)\ \frac{\partial S_N}{\partial x_N} = 0 \Rightarrow \underbrace{v_N'(\hat{x}_N)}_{\text{marginal social benefit}} = \underbrace{C'(\hat{x}_N)}_{\text{marginal production cost}} + \underbrace{I_N\gamma_N'(I_N x_N + Z_S)}_{\text{marginal North externality}}$$

3.8 Social (World) Surplus

$$\max_{x_S, x_N} S(x_S, x_N) = I_S[v_S(x_S) - C(x_S) - \gamma(I_S x_S + I_N x_N)] + I_N[v_N(x_N) - C(x_N) - \gamma(I_S x_S + I_N x_N)]$$

$$(1)\ \frac{\partial S}{\partial x_S} = 0 \Rightarrow \underbrace{v_S'(x_S^*)}_{\text{marginal world benefit}} = \underbrace{C'(x_S^*)}_{\text{marginal production cost}} + \underbrace{I_S\gamma_S'(I_S x_S^* + I_N x_N^*)}_{\text{marginal South externality}} + \underbrace{I_N\gamma_N'(I_S x_S^* + I_N x_N^*)}_{\text{marginal North externality}}$$

$$(2)\ \frac{\partial S}{\partial x_N} = 0 \Rightarrow \underbrace{v_N'(x_N^*)}_{\text{marginal world benefit}} = \underbrace{C'(x_N^*)}_{\text{marginal production cost}} + \underbrace{I_N\gamma_N'(I_S x_S^* + I_N x_N^*)}_{\text{marginal North externality}} + \underbrace{I_S\gamma_S'(I_S x_S^* + I_N x_N^*)}_{\text{marginal South externality}}$$

3.9 Abatement

α = abatement (amount of externality reduced)
$Z = x_B - \alpha$
$\xi(\alpha)$ = cost of abatement

3.9.1 Maximization of Social Surplus

$$\max_{x_B, x_R, \alpha} S(x_B, x_R, \alpha) = v_B(x_B) + v_R(x_R) - C(x_B + x_R) - \gamma(x_B - \alpha) - \xi(\alpha)$$

$$(1)\ \frac{\partial S}{\partial x_B} = 0 \Rightarrow \underbrace{v_B'(x_B^*)}_{\text{marginal social benefit}} = \underbrace{\underbrace{C'(x_B^* + x_R^*)}_{\text{marginal production cost}} + \underbrace{\gamma'(x_B^* - \alpha^*)}_{\text{marginal external cost}}}_{\text{marginal social cost}}$$

$$(2)\ \frac{\partial S}{\partial x_R} = 0 \Rightarrow \underbrace{v_R'(x_R^*)}_{\text{marginal social benefit}} = \underbrace{C'(x_B^* + x_R^*)}_{\text{marginal social cost}}$$

$$(3)\ \frac{\partial S}{\partial \alpha} = 0 \Rightarrow \underbrace{-\gamma'(x_B^* - \alpha^*)}_{\text{marginal externality reduction}} = \underbrace{\xi'(\alpha^*)}_{\text{marginal abatement cost}}$$

3.10 Bargaining

Disagreement Point The (typically) inefficient starting point at which Blue and Red fall if they do not agree to a more efficient solution.

Complete Bargaining When all opportunities for Pareto improvements through bargaining are taken.

When there are externalities, an efficient outcome can be achieved through bargaining if there is complete bargaining and no transaction costs.

4 Public Goods

Nonrivalness The amount of a good available to a person does not decrease when an extra person uses the good.

Excludability The ability to prevent someone from using a good.

Free Disposal The ability to consume less of a good than is made available to you.

If there is free disposal then $v_i'(x) = \begin{cases} v_i'(x) & \text{if } U_i(x) \geq 0 \\ 0 & \text{if } U_i(x) < 0 \end{cases}$

x = the amount of public good available
x_i = the amount of public good available to person i
$\tilde{x}_i$ = the amount of public good demanded by person i

$\tilde{x}_i \leq x_i \leq x$
If good X is nonexcludable then $x_i = x$.
If there is no free disposal then $\tilde{x}_i = x_i$.

$S(x) = \sum_{i=1}^{I} v_i(x) - C(x)$

4.1 Maximization of Social Surplus

$$\max_x S(x) = \sum_{i=1}^{I} v_i(x) - C(x)$$

$$(1)\, \frac{\partial S}{\partial x} = 0 \Rightarrow \underbrace{\sum_{i=1}^{I} v_i'(x)}_{\text{total marginal valuation}} = \underbrace{C'(x)}_{\text{marginal cost}} \quad \textbf{Samuelson Condition}$$

4.2 Lindahl Equilibrium

s_R = share paid by the rich
s_P = share paid by the poor
I_R = number of rich people
I_P = number of poor people
$\omega = I_R\omega_R + I_P\omega_P$
A Linddahl equilibrium is two cost shares, s_R and s_P, that add up too 1 and that induce the rich and the poor to propose the same quantity of public good.

4.2.1 UMAX Rich

Maximize $U_R(x, m_R) = \sqrt{v_R(x) + m_R}$ subject to $\frac{s_R C(x)}{I_R} + m_R = \omega_R$

$$\max_x v_R(x) + \omega_R - \frac{s_R C(x)}{I_R}$$

$$(1) \frac{\partial U_R}{\partial x} = 0 \Rightarrow s_R = \frac{I_R v_R'(x)}{C'(x)}$$

4.2.2 UMAX Poor

Maximize $U_P(x, m_P) = \sqrt{v_P(x) + m_P}$ subject to $\frac{s_P C(x)}{I_P} + m_P = \omega_P$

$$\max_x v_P(x) + \omega_P - \frac{s_P C(x)}{I_P}$$

$$(1) \frac{\partial U_P}{\partial x} = 0 \Rightarrow s_P = \frac{I_P v_P'(x)}{C'(x)}$$

4.2.3 Equilibrium

A Linddahl equilibrium is two cost shares, s_R and s_P, that add up to 1 and that induce the rich and the poor to propose the same quantity of public good.

4.3 Taxes and Voting

$t_i(x)$ = tax used to finance the public good
$\beta_i(x) = v_i(x) - t_i(x)$ net benefit function

Single Peaked Preferences If you take two points on the same side of a consumer's peak net benefit, then the consumer prefers the point closer to the peak.

Median Voter A median voter exists if the median of the distribution of peak utility values is someone's personal peak.

Median Voter Theorem If net benefits are single peaked and a median voter exists, then their preference will win a majority vote (if there is no abstention).

4.4 Public Contribution

$t_i = i$'s personal contribution to the public good
$C(x) = x$
$x = \sum_{h=1}^{I} t_h$

4.4.1 UMAX i

Maximize $U_i(x, m_i) = \sqrt{v_i(t_i + \sum_{h \neq i}^{I} \tilde{t}_h) + m_i}$ subject to $t_i + m_i = \omega_i$, $t_i \geq 0$

$$\max_{t_i} v_i(t_i + \sum_{h \neq i}^{I} \tilde{t}_h) + \omega_i - t_i$$

$$(1) \frac{\partial U_i}{\partial t_i} = 0 \Rightarrow \begin{cases} v_i'(\tilde{t}_i + \sum_{h \neq i}^{I} \tilde{t}_h) = 1 & \text{if } t_i > 0 \\ v_i'(\tilde{t}_i + \sum_{h \neq i}^{I} \tilde{t}_h) \leq 1 & \text{if } t_i = 0 \end{cases}$$

4.4.2 Cournot-Nash Equilibrium

Cournot-Nash Equilibrium Each player is doing the best they can for themselves, taking what everyone else is doing as given.

The strategy chosen by any given player yields a higher utility than any other strategy available to her, when all the other players play their equilibrium strategies.

⇒ No regret

Stand-Alone Contribution The amount $\hat{t}_i$ that person i contributes if no one else contributes to the public good.

$$\begin{cases} v_i'(\hat{t}_i) = 1 & \text{if } \hat{t}_i > 0 \\ v_i'(\hat{t}_i) \leq 1 & \text{if } \hat{t}_i = 0 \end{cases}$$

Steps to find the Cournot-Nash Equilibrium of a voluntary contribution game:

1. Find the stand-alone contribution ($\hat{t}_i$) of each player ($i = 1, \ldots, I$).
2. Find the player or players with the largest stand-alone contribution (the *top valuators*), and define $\hat{t}$ as the largest out of all the $\hat{t}_i$.
3. The level of the public good at any Cournot-Nash equilibrium equals the largest stand-alone contribution, $\hat{t}$. Players who are not top valuators contribute nothing and *free ride* on whoever contributes.

4. If there are multiple top valuators, then any contributions by the top valuators that add up to $\hat{t}$ is an equilibrium.

The voluntary contribution equilibrium is *inefficient* because it leads to not enough of the public good.
$\sum_{i=1}^{I} v_i'(\hat{t}) > 1$ **Voluntary Contribution**
$\sum_{i=1}^{I} v_i'(x) = 1$ **Samuelson Condition**

4.5 User Fees

If exclusion is possible then we can charge user fees to access the public good.

4.5.1 UMAX i

Maximize $U_i = \sqrt{v_i(x) + m_i}$ subject to $p_i x + m_i = \omega_i$

$$\max_x v_i(x) + \omega_i - p_i x$$

$$\frac{\partial U_i}{\partial x} = 0 \Rightarrow v_i'(\tilde{x}) = p_i$$

Efficient if $p_i^* = v_i'(x^*)$

5 Public Utilities

$k =$ capacity
$\varphi =$ cost per unit of capacity
$\tilde{C}(k, x_B, x_R) = \underbrace{\varphi k}_{\text{Capacity Cost}} + \underbrace{c_B x_b + c_R x_R}_{\text{Operating Cost}}$
$x_B \leq k$ Blue cannot consume more than the capacity
$x_R \leq k$ Red cannot consume more than the capacity

Blue and Red purchase their goods at two different times, so the capacity is nonrival. Blue's consumption does not count against the capacity of Red's consumption.

$k = \max\{x_B, x_R\}$ There is no need to build excess capacity if there are costs to capacity
Due to free disposal, $\hat{v}_i'(k) = \max\{v_i'(k) - c_i, 0\} \quad i = B, R$

5.1 Maximization of Social Surplus

$$\max_{k, x_B, x_R} S(k) = v_B(k) + v_R(k) - \varphi k - c_B x_B - c_R x_R$$

$$(1) \frac{\partial S}{\partial k} = 0 \Rightarrow \underbrace{v_B'(k^*) + v_R'(k^*)}_{\text{total marginal valuation}} = \underbrace{\varphi}_{\text{marginal cost}} \quad \textbf{Samuelson Condition}$$

$$(2)\frac{\partial S}{\partial x_B} = 0 \Rightarrow \underbrace{v_B'(x_B^*)}_{\text{marginal valuation}} = \underbrace{\begin{cases} \varphi + c_B & \text{if } x_B^* = k^* \\ c_B & \text{if } x_B^* \leq k^* \end{cases}}_{\text{marginal cost}}$$

$$(3)\frac{\partial S}{\partial x_R} = 0 \Rightarrow \underbrace{v_R'(x_R^*)}_{\text{marginal valuation}} = \underbrace{\begin{cases} \varphi + c_R & \text{if } x_R^* = k^* \\ c_R & \text{if } x_R^* \leq k^* \end{cases}}_{\text{marginal cost}}$$

5.2 Lindahl Equilibrium for Capacity

5.2.1 UMAX Blue

Maximize $U_B(x, m_B) = \sqrt{v_B(k) + m_B}$ subject to $\varphi_B k + m_B = \omega_B$

$$\max_k v_B(k) + \omega_B - \varphi_B k$$

$$(1)\frac{\partial U_B}{\partial k} = 0 \Rightarrow \underbrace{v_B'(k^*)}_{\text{marginal valuation}} = \underbrace{\varphi_B}_{\text{marginal private cost}}$$

Blue pays her marginal valuation of the total capacity (which may be 0 if k is large enough).

5.2.2 UMAX Red

Maximize $U_R(x, m_R) = \sqrt{v_R(k) + m_R}$ subject to $\varphi_R k + m_R = \omega_R$

$$\max_k v_R(k) + \omega_R - \varphi_R k$$

$$(1)\frac{\partial U_R}{\partial k} = 0 \Rightarrow \underbrace{v_R'(k^*)}_{\text{marginal valuation}} = \underbrace{\varphi_R}_{\text{marginal private cost}}$$

Red pays his marginal valuation of the total capacity (which may be 0 if k is large enough).

5.2.3 Efficiency

$\varphi_B + \varphi_R = \varphi \Rightarrow v_B'(k^*) + v_R'(k^*) = \varphi$ **Samuelson Condition**

5.3 Peak-Load Pricing

5.3.1 UMAX Blue

Maximize $U_B(x, m_B) = \sqrt{v_B(k) + m_B}$ subject to $p_B[k + x_B] + m_B = \omega_B$

$$\max_{x_B} v_B(x_B) + \omega_B - p_B[k + x_B]$$

$$(1) \frac{\partial U_B}{\partial x_B} = 0 \Rightarrow \underbrace{v_B'(\tilde{x}_B)}_{\text{marginal valuation}} = \underbrace{p_B}_{\text{marginal private cost}}$$

Efficient if $p_B^* = \varphi_B + c_B$

5.3.2 UMAX Red

Maximize $U_R(x, m_R) = \sqrt{v_R(k) + m_R}$ subject to $p_R[k + x_R] + m_R = \omega_R$

$$\max_{x_R} v_R(x_R) + \omega_R - p_R[k + x_R]$$

$$(1) \frac{\partial U_R}{\partial x_R} = 0 \Rightarrow \underbrace{v_R'(\tilde{x}_R)}_{\text{marginal valuation}} = \underbrace{p_R}_{\text{marginal private cost}}$$

Efficient if $p_R^* = \varphi_R + c_R$

5.4 Natural Monopoly

$C(x) = \begin{cases} 0 & \text{if } x = 0 \\ F + cx & \text{if } x > 0 \end{cases}$

Average Cost $= \begin{cases} 0 & \text{if } x = 0 \\ \frac{F}{x} + c & \text{if } x > 0 \end{cases}$

$S(x_B, x_R) = \begin{cases} 0 & \text{if } x_B = x_R = 0 \\ v_B(x_B) + v_R(x_R) - F - cx_B - cx_R & \text{if } x_B + x_R > 0 \end{cases}$

Viability Condition Given the fixed costs, the social surplus is positive for some nonzero $(x_B, x_R) \Rightarrow x_B^*, x_R^* > 0$

5.4.1 Maximization of Social Surplus

$$\max_{x_B, x_R} v_B(x_B) + v_R(x_R) - F - cx_B - cx_R$$

$$(1) \frac{\partial S}{\partial x_B} = 0 \Rightarrow v_B'(x_B^*) = c$$

$$(2) \frac{\partial S}{\partial x_R} = 0 \Rightarrow v_R'(x_R^*) = c$$

5.4.2 UMAX Blue

Maximize $U_B(x_B, m_B) = \sqrt{v_B(x_B) + m_B}$ subject to $px_B + m_B = \omega_B$

$$\max_{x_B} U_B(x_B) = v_B(x_B) + \omega_B - px_B$$

$$(1) \frac{\partial U_B}{\partial x_B} = 0 \Rightarrow \underbrace{v_B'(\tilde{x}_B)}_{\text{marginal valuation}} = \underbrace{p}_{\text{marginal private cost}}$$

5.4.3 UMAX Red

Maximize $U_R(x_R, m_R) = \sqrt{v_R(x_R) + m_R}$ subject to $px_R + m_R = \omega_R$

$$\max_{x_R} U_R(x_R) = v_R(x_R) + \omega_R - px_R$$

$$(1)\ \frac{\partial U_R}{\partial x_R} = 0 \Rightarrow \underbrace{v_R'(\tilde{x}_R)}_{\text{marginal valuation}} = \underbrace{p}_{\text{marginal private cost}}$$

5.4.4 First and Second Best Solution

Efficient if $p^* = c$
However, because marginal cost < average cost, this will cause the firm to lose money. The firm will make c on every unit that they sell, but it will cost them $c + \frac{F}{x}$ to produce each unit. So the firm will cover all of their marginal costs but not their fixed costs, and they will have negative profits equal to their fixed cost F.
$\Rightarrow$ If prices are restricted to be linear, then efficiency implies losses for the firm.

Second-Best Solution The highest surplus consistent with nonnegative profits.

The second best solution is where the average cost curve intersects the demand curve. The firm makes 0 profits and sells a less-than-efficient quantity. This creates deadweight loss due to the fact that there are consumers who are willing to pay more than the marginal cost for an additional unit of x.

5.4.5 Two-Part Tariffs

A_i = *access fee* to consume the good
p_i = *marginal price* or *marginal fee* per unit
Consumer i pays $\begin{cases} 0 & \text{if } x_i = 0 \\ A_i + p_i x_i & \text{if } x_i > 0 \end{cases}$
Efficient if $p_i^* = c \Rightarrow$ the firm will make negative profits equal to F.

Break-Even Condition The firm must make nonnegative profits or they will not produce.

$A_B + A_R = F$

Participation Constraint The consumer will only participate if they are better off participating than not participating.

$CS_i^* = v_i(x_i^*) - cx_i^* \geq A_i \quad i = B, R$

There is an efficient allocation where the firm breaks even if
$CS_B^* + CS_R^* \geq A_B + A_R = F \Rightarrow CS_B^* + CS_R^* \geq F$